On Abelianized Absolute Galois Groups of Global Fields

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Introduction

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Moreover, the norm map $\mathbb{C}^\times \to \mathbb{R}^\times$, which sends z to $|z|$ gives us the following exact sequence :

$$1 \to S_1 = \{z \in \mathbb{C} | |z| = 1\} \to \mathbb{C}^\times \to \mathbb{R}^\times \to \mathbb{Z}/2\mathbb{Z} \to 1$$

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One of the main theorems of class field theory says that for any finite *abelian* extensions of local fields L/K with Galois group G there exists a norm map N from $L^\times$ to $K^\times$ and the following isomorphism holds:

$$K^\times / N(L^\times) \simeq G$$

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Theorem (Artin-Schreier)

Suppose G_K is finite. Then

1. $G_K \simeq \mathbb{Z}/2\mathbb{Z}$;
2. *K has characteristic zero;*
3. *K is real closed field, i.e. $\overline{K} = K(i)$ where $i^2 = -1$;*

The Absolute Galois of a Finite Field

Let us consider a few more examples. Let $K = \mathbb{F}_q$ where $q = p^n$, p is prime. Any finite extension L of K is abelian with cyclic Galois group generated by the Frobenious $Fr_K : x \to x^{p^n}$. One has:

$$G_K \simeq \hat{\mathbb{Z}},$$

where $\hat{\mathbb{Z}}$ denotes the group of pro-finite integers:

$$\hat{\mathbb{Z}} = \{(a_n) \in \prod_{n=1}^{\infty} (\mathbb{Z}/n\mathbb{Z}) | \forall n, m : n|m \Rightarrow a_m = a_n \mod n\}.$$

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It is convenient to represent any element $x \in \hat{\mathbb{Z}}$ as infinite series in the factorial base:

$$x = a_1 1! + a_2 2! + a_3 3! + \ldots a_n n! + \ldots,$$

where $0 \leq a_n \leq n$.

Properties of $\hat{\mathbb{Z}}$

Some properties of $\hat{\mathbb{Z}}$.

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4. $\hat{\mathbb{Z}}$ is a compact, Hausdorff totally disconnected topological ring;
5. $A \times \hat{\mathbb{Z}} \simeq A$ as abelian groups, but not as topological groups.

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Definition

A topological group G is pro-finite if one of the following equivalent conditions holds:

1. *G is a compact, Hausdorff totally disconnected topological group;*
2. *G is isomorphic to a closed subgroup of a product of finite discrete groups*
3. *G is isomorphic to the inverse limit of an inverse system of discrete finite groups.*

Note that to any group G one associates its pro-finite completion $\tilde{G} = \lim_N (G/N)$, where N runs over normal subgroups of finite index.

Neukirch-Uchida

Definition

A global field K is:

1. *either a number field i.e. finite extension of $\mathbb{Q}$; (char$(K) = 0$)*
2. *or a function field i.e. field of functions on smooth projective curve over finite field $\mathbb{F}_q$; (char$(K) = p > 0$)*

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Theorem

Suppose K, K' are global fields such that $G_K \simeq G_{K'}$ as topological groups. Then $K \simeq K'$ as fields(up to Frobenious twist in the case of positive characteristic).

Jurgen Neukirch (24 July 1937 – 5 February 1997)

He gave a proof for the case of normal extensions of $\mathbb{Q}$ in 1969. Then Uchida (and also Ikeda, Iwasawa) extended his results in 1976 to arbitrary number and function fields.
Wikipedia :" *The Neukirch–Uchida theorem is one of the foundational results of anabelian geometry, whose main theme is to reduce properties of geometric objects to properties of their fundamental groups, provided these fundamental groups are sufficiently non-abelian.*"

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Today I will explain why it is not enough to consider G_K^{ab} itself.

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Today I will explain why it is not enough to consider G_K^{ab} itself. Class field theory provides a nice description of G_K^{ab} in terms of other "inner" invariants of K, but to describe the isomorphism type G_K^{ab} more or less explicitly requires a lot of work.

Class Field Theory for $\mathbb{Q}$

Let us consider example $K = \mathbb{Q}$. An easy way to construct abelian extensions of $\mathbb{Q}$ is to adjoin n-th root of unity $\zeta_n \in \mathbb{C} : z^n = 1$. The field $K_n = \mathbb{Q}(\zeta_n)$ is abelian extension of $\mathbb{Q}$ with Galois group isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{\times}$.

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Example

Let $L = \mathbb{Q}(\sqrt{5})$ this is abelian extension with Galois group C_2. It is easy to see that $L \subset \mathbb{Q}(\zeta_5)$. Let $L = \mathbb{Q}(\alpha)$ where α is a root of $x^3 - 3x + 1$. This is abelian extension with Galois group C_3. One has $L \subset \mathbb{Q}(\zeta_{81})$.

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If $l \neq 2$ then:

$$\mathbb{Z}_l^{\times} \simeq \mathbb{F}_l^{\times} \times \mathbb{Z}_l$$

For $l = 2$ one has:

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Hence:

$$G_{\mathbb{Q}}^{ab} \simeq \hat{\mathbb{Z}} \times T_{\mathbb{Q}},$$

Where $T_{\mathbb{Q}} = \mathbb{Z}/2\mathbb{Z} \times \prod_{l \neq 2} \mathbb{Z}/(l-1)\mathbb{Z}$.

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Where $T_{\mathbb{Q}} = \mathbb{Z}/2\mathbb{Z} \times \prod_{l \neq 2} \mathbb{Z}/(l-1)\mathbb{Z}$. By the Dirichlet's theorem on arithmetic progressions one has:

$$T_{\mathbb{Q}} \simeq \prod_n (\mathbb{Z}/n\mathbb{Z}).$$

$T_{\mathbb{Q}}$ is the closure of the torsion subgroup of $G_{\mathbb{Q}}^{ab}$.

Now it is natural to ask how we could generalise the above picture. Unfortunately for any other number field K we don't have a good explicit description of the maximal abelian extension of K. But the global class field theory still provides us with the description of G_K^{ab} in terms of the *Idele class group* of K.

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By definition $C_K = (\prod'_v K_v^\times)/K^\times$, where v runs over places of K, K_v denotes the corresponding completion, $K^\times$ is embedded diagonally and for given $x \in \prod'_v K_v^\times$ almost all $x_v \in \mathcal{O}_v^\times$.

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$$C_K \to G_K^{ab} \to 1$$

And kernel is given in terms of infinite places of K.

Let K be an Imaginary quadratic field i.e. $K = \mathbb{Q}(d)$, where $d < 0$ is a square-free integer. In this case one has:

$$G_K^{ab} \simeq \left(\prod_{v \text{ is finite}}' K_v^\times \right) / K^\times .$$

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There exists also an exact sequence:

$$1 \to \mathcal{O}_K^{\times} \to \prod_{v \text{ is finite}} \mathcal{O}_v^{\times} \to G_K^{ab} \to \mathrm{Cl}(K) \to 1$$

Where $\mathrm{Cl}(K)$ is the ideal class group of K.

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Where $\mathrm{Cl}(K)$ is the ideal class group of K.
By the Dirichlet's units theorem the unit group $\mathcal{O}_K^\times$ has rank 0.
There are three possibilities for this torsion group $\mathcal{O}_K^\times$: ± 1, C_4, C_6.
This is always ± 1, except two cases where K is either $\mathbb{Q}(i)$ or $\mathbb{Q}(\zeta_6)$.

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This is always ± 1, except two cases where K is either $\mathbb{Q}(i)$ or $\mathbb{Q}(\zeta_6)$.
It turned out that if $K \neq \mathbb{Q}(i)$, $\mathbb{Q}(\zeta_6)$ then

$$(\prod_{v \text{ is finite}} \mathcal{O}_v^\times)/\mathcal{O}_K^\times \simeq T_{\mathbb{Q}} \times \hat{\mathbb{Z}}^2$$

It means that we could write the above exact sequence as:

$$1 \to \mathcal{T} \times \hat{\mathbb{Z}}^2 \to G_K^{ab} \to \mathrm{Cl}(K) \to 1,$$

here $\mathcal{T} = T_{\mathbb{Q}}$ is the closure of the torsion subgroup of G_K^{ab}.

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Corollary (Onabe, 1976)

Suppose K is an imaginary quadratic field with $\mathrm{Cl}(K) = 1$ different from $\mathbb{Q}(i)$, $\mathbb{Q}(\zeta_6)$. Then $G_K^{ab} \simeq \mathcal{T} \times \hat{\mathbb{Z}}^2$.

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Theorem (Angelakis and Stevenhagen, 2013)

Let K be as before imaginary quadratic field different from $\mathbb{Q}(i)$, $\mathbb{Q}(\zeta_6)$. Then:

1. *The topological closure $\overline{\mathcal{G}_K^{ab}[tor]}$ of the torsion subgroup of $\mathcal{G}_K^{ab}$ is $\mathcal{T}$;*
2. *The torsion subgroup of the quotient $\mathcal{G}_K^{ab}/\mathcal{T}$ is trivial if and only if $\mathcal{G}_K^{ab} \simeq \widehat{\mathbb{Z}}^2 \times \mathcal{T}$;*
3. *There exist an injective map from $(\mathcal{G}_K^{ab}/\mathcal{T})[tor]$ to $\mathrm{Cl}(K)$ and an algorithm with input K and output whether the group $(\mathcal{G}_K^{ab}/\mathcal{T})[tor]$ is trivial or not.*

According to their computations almost all K have $G_K^{ab} \simeq \mathcal{T} \times \hat{\mathbb{Z}}^2$, but they can't prove there are infinitely many such fields.

Motivated by the above results authors of the present paper started working on the question about isomorphism type of $\mathcal{G}_K^{ab}$ where K denotes a global function field. Our technique allows us also to improve their results on imaginary quadratic fields: we constructed infinitely many fields with non-isomorphic $\mathcal{G}_K^{ab}$ and also many new examples with isomorphic $\mathcal{G}_K^{ab} \not\simeq \mathcal{T} \times \hat{\mathbb{Z}}^2$.

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For a global function field K of characteristic p with the exact constant field $\mathbb{F}_q$, $q = p^n$ we defined the invariant d_K as a natural number such that $n = p^k d_K$ with $\gcd(d_K, p) = 1$. Let $\mathrm{Cl}^0(K)$ denotes the degree zero part of the class-group of K. In other words $\mathrm{Cl}^0(K)$ is the abelian group of $\mathbb{F}_q$-rational points of the Jacobian variety associated to the curve X.

Theorem

Suppose K and K' are two global function fields, then $\mathcal{G}_K^{ab} \simeq \mathcal{G}_{K'}^{ab}$ as pro-finite groups if and only if the following three conditions hold:

1. *K and K' share the same characteristic p;*
2. *Invariants d_K and $d_{K'}$ coincide: $d_K = d_{K'}$;*
3. *The non p-parts of class-groups of K and K' are isomorphic:*

$$\mathrm{Cl}^0_{non-p}(K) \simeq \mathrm{Cl}^0_{non-p}(K').$$

In particular, two function fields with the same exact constant filed $\mathbb{F}_q$ have isomorphic $\mathcal{G}_K^{ab}$ if and only if they have isomorphic $\mathrm{Cl}^0_{non-p}(K)$.

Corollary

Let K be the rational function field (with genus zero) over fixed constant field $\mathbb{F}_q$ and let E be an elliptic function field (with genus one) defined over the same constant field, such that $\# \mathrm{Cl}^0(E) = q$. Then there exists isomorphism of topological groups $\mathcal{G}_K^{ab} \simeq \mathcal{G}_E^{ab}$. In particular, the genus g of K and therefore the Dedekind zeta-function $\zeta_K(s)$ of K are not determined by $\mathcal{G}_K^{ab}$ even if the constant field $\mathbb{F}_q$ is fixed.

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Corollary

There are infinitely many function fields of the same characteristic p(but with different cardinality of the constant field) with isomorphic $\mathcal{G}_K^{ab}$.

Outline of the Proof

As before the Artin map provides us with the homomorphism $\mathcal{C}_K \to \mathcal{G}_K^{ab}$, where $\mathcal{C}_K$ is the Idele class group of K. In the function field case this map is injective, but not surjective, since $\mathcal{C}_K$ is not compact.

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But if we take the pro-finite completion(with respect to the given topology) of $\mathcal{C}_K$ then the main theorem of the class field theory says that we have isomorphism of topological groups:

$$\widehat{\mathcal{C}_K} \simeq \mathcal{G}_K^{ab}$$

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Recall that we have a split exact sequence:

$$0 \to \mathcal{C}_K^0 \to \mathcal{C}_K \to \mathbb{Z} \to 0,$$

where $\mathcal{C}_K^0$ is the degree zero part of the Idele class group and the map from $\mathcal{C}_K$ to $\mathbb{Z}$ is the degree map. Now, $\mathcal{C}_K^0$ is pro-finite, hence complete and therefore $\widehat{\mathcal{C}_K} \simeq \mathcal{C}_K^0 \oplus \widehat{\mathbb{Z}}$

Step one is the folloiwng:

Lemma

Let A and B be two pro-finite abelian groups. Then $A \simeq B$ if and only if $A \oplus \widehat{\mathbb{Z}} \simeq B \oplus \widehat{\mathbb{Z}}$ in the category of pro-finite abelian groups.

This lemma reduces our question to the description of $\mathcal{C}_K^0$ as topological group.

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Step two:

Lemma

There exists an exact sequence of topological groups, where all finite groups treated with the discrete topology:

$$1 \to \mathbb{F}_q^\times \to \prod_v \mathcal{O}_v^\times \to \mathcal{C}_K^0 \to \mathrm{Cl}^0(K) \to 1.$$

Outline of the Proof

By using the following isomorphism $\mathcal{O}_v^\times \simeq \mathbb{F}_{q^n}^\times \times \mathbb{Z}_p^\infty$, where n is a degree of v we reduce the above sequence:

$$1 \to \mathcal{T}_K \times \mathbb{Z}_p^\infty \to \mathcal{C}_K^0 \to \mathrm{Cl}^0(K) \to 1,$$

where $\mathcal{T}_K \simeq (\prod_v \mathbb{F}_{q^{\deg(v)}}^\times)/\mathbb{F}_q^\times$.

Lemma

Two groups $\mathcal{T}_K$ and $\mathcal{T}_{K'}$ are isomorphic if and only if $p = p'$ and $d_K = d_{K'}$.

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Lemma

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Our next step is to show that $\mathcal{T}_K$ is the topological closure of the torsion subgroup of $\mathcal{C}_K^0$:

Lemma

Image of any torsion element x of $\mathcal{C}_K^0$ in $\mathrm{Cl}^0(K)$ is zero.

Outline of the Proof

I.e. we've proved:

Theorem

Suppose K, K' *are two function fields such that* $G_K^{ab} \simeq G_{K'}^{ab}$. *Then* $p = p'$ *and* $d_K = d_{K'}$.

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In order to recover $\mathrm{Cl}^0_{non-p}(K)$ we need the following observation. For any prime number $l \neq p$ we have:

$$1 \to \mathcal{T}_{K,l} \to \mathcal{C}^0_{K,l} \to \mathrm{Cl}^0_l(K) \to 1,$$

i.e.

$$\mathcal{C}^0_{K,l} \,/\, \mathcal{C}^0_{K,l}[tors] \simeq \mathrm{Cl}^0_l(K)$$

And hence for any $l \neq p$ we have: if $G_K^{ab} \simeq G_{K'}^{ab}$ then $\mathrm{Cl}^0_l(K) \simeq \mathrm{Cl}^0_l(K')$.

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Suppose K, K' are two function fields such that $G_K^{ab} \simeq G_{K'}^{ab}$. Then $p = p'$ and $d_K = d_{K'}$.

In order to recover $\mathrm{Cl}^0_{non-p}(K)$ we need the following observation. For any prime number $l \neq p$ we have:

$$1 \to \mathcal{T}_{K,l} \to \mathcal{C}^0_{K,l} \to \mathrm{Cl}^0_l(K) \to 1,$$

i.e.

$$\mathcal{C}^0_{K,l} \,/\, \mathcal{C}^0_{K,l}[tors] \simeq \mathrm{Cl}^0_l(K)$$

And hence for any $l \neq p$ we have: if $G_K^{ab} \simeq G_{K'}^{ab}$ then $\mathrm{Cl}^0_l(K) \simeq \mathrm{Cl}^0_l(K')$.
Therefore we proved the *only if* part. The rest is slightly more complicated.

I.e. we've proved:

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Coffee Break!

Thank you!