

# On Artin L-functions and Gassmann Equivalence for Global Function Fields

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Let  $K$  and  $L$  be two number fields. We will say that they *split equivalently* if for any prime number  $p \in \mathbb{Z}$  there exists a bijection  $\phi_p$  from the set of primes in  $\mathcal{O}_K$  lying above  $p$  to the set of those primes in  $\mathcal{O}_L$ . We will say that they are *arithmetically equivalently* if for each  $p$  the bijection  $\phi_p$  is degree preserving.

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Let  $N$  denote the common Galois closure of  $K$  and  $L$  over  $\mathbb{Q}$  and let  $G = \text{Gal}(N/\mathbb{Q})$ ,  $H = \text{Gal}(N/K)$ ,  $H' = \text{Gal}(N/L)$ . We will call a triple  $(G, H, H')$  a *Gassmann triple* if we have isomorphism of induced representation  $\text{Ind}_H^G(1_H) = \text{Ind}_{H'}^G(1_{H'})$ , where  $1_H$  (and  $1_{H'}$ ) means trivial representation of  $H$  (of  $H'$  respectively).

Then we have the following famous result

## Theorem (Perlis)

*The following statements are equivalent:*

- 1  $\zeta_K(s) = \zeta_L(s)$ ;
- 2  $K$  and  $L$  are arithmetically equivalently;
- 3  $K$  and  $L$  split equivalently;
- 4  $(G, H, H')$  form a Gassmann triple.

This theorem allows us to use group theory in order to study some number-theoretical relations between  $K$  and  $L$ .

For example,

## Theorem (Perlis, Dipasquale)

*Let  $G$  be a finite group and  $H \subset G$  a subgroup of index  $n$ .  
Suppose one of the following conditions holds:*

- 1  $n \leq 6$ ;
- 2  $H$  is cyclic;
- 3  $G = \mathbb{S}_n$  the full symmetric group of order  $n$ ;
- 4 Order of  $G$  is divisible by at most four not necessarily different primes.

*then any Gassmann triple  $(G, H, H')$  is trivial and therefore  $\zeta_K(s)$  determines the number field  $K$  up to isomorphism!*

The same time there are a lot of examples of non-isomorphic number fields  $K, L$  sharing the same zeta-function.

## Example

*Fix an integer  $a$ , such that neither  $|a|$  nor  $2|a|$  are squares. Then fields  $\mathbb{Q}(a^{\frac{1}{8}})$  and  $\mathbb{Q}((16a)^{\frac{1}{8}})$  are non-isomorphic arithmetically equivalent number fields.*

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By using group theory one has the following:

## Theorem (Bart de Smit)

*For any given number field  $K$  there exists an abelian character  $\chi$  such that  $L_K(\chi, s)$  determines field  $K$  up to isomorphism.*

Could we generalise this to the function field side?

Let  $X, X'$  be two curves over  $k = \mathbb{F}_q$ ,  $q = p^n$ . Fix  $k$ -rational separable maps from  $X$  and  $X'$  to  $\mathbb{P}^1$ . In other words, consider two geometric separable extensions of  $\mathbb{F}_q(x)$ . Then by analogy we have a notion of arithmetical, splitting and gassmann equivalence.

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**Observation:** These three notions are still equivalent. But equality of dedekind zeta-functions is weaker.

## Example

Consider two elliptic curves  $E$  and  $E'$  over  $\mathbb{F}_7$ , affine part of which defined by equations  $y^2 = x^3 + 1$  and  $y^2 = x^3 + 3x + 1$  respectively. Let us denote by  $K$  and  $K'$  corresponding function fields. One checks that

$$\zeta_K(T) = \frac{7T^2 + 4T + 1}{(1 - T)(1 - 7T)} = \zeta_{K'}(T),$$

where  $T = 7^{-s}$ . Hence by the theorem of Weil,  $E$  and  $E'$  are  $\mathbb{F}_7$ -isogenous, but  $j(E) = 0$  and  $j(E') = 2$  so they are not isomorphic even over the algebraic closure  $\overline{\mathbb{F}_7}$  and hence  $K \not\cong K'$ .

**Observation:** Consider the common normal closure  $N$ . We have  $G \simeq C_2 \oplus C_2 \simeq H \oplus H'$  and there exists abelian character  $\chi$  of  $G$  such that  $\chi|_H = 1$  and  $\chi|_{H'} \neq 1$ . But then  $L_K(\chi|_H) = \zeta_K$  and has pole at  $s = 1$ . The same time  $L_{K'}(\chi|_{H'})$  is an  $L$ -function of a non-trivial abelian character and therefore has no poles!

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Motivated by this observation K.Nagata in 1976 published a paper "On Artin L-functions and Gassmann Equivalence". Today we will discuss further generalizations of Nagata's results.

## Theorem

Let  $K, K'$  denote two finite separable geometric extensions of  $\mathbb{F}_q(x)$ . Let  $N$  denote the common Galois closure and  $G = \text{Gal}(N/\mathbb{F}_q(x))$ ,  $H = \text{Gal}(N/K)$ ,  $H' = \text{Gal}(N/K')$ . Let  $\alpha$  denote a complex representation of  $H$  and  $\alpha'$  denote a complex representation of  $H'$ . Let  $\rho_1, \dots, \rho_n$  denote all irreducible complex representations of  $G$  and  $\bar{\rho}$  denotes the dual representation of  $\rho$ . Let  $\psi = \text{Ind}_H^G(\alpha)$  and  $\psi' = \text{Ind}_{H'}^G(\alpha')$ . The following are equivalent:

- 1 For all  $i$  such that  $1 \leq i \leq n$  we have equality of Artin  $L$ -functions:  $L_K(\alpha \otimes \rho_i|_H) = L_{K'}(\alpha' \otimes \rho_i|_{H'})$
- 2  $L_K(\bar{\alpha} \otimes (\psi|_H)) = L_{K'}(\bar{\alpha}' \otimes (\psi|_{H'}))$  and  $L_K(\bar{\alpha} \otimes (\psi'|_H)) = L_{K'}(\bar{\alpha}' \otimes (\psi'|_{H'}))$ ;
- 3 Induced representations  $\psi$  and  $\psi'$  are isomorphic.

**from (1) to (3).** For any fixed representation  $\rho$  of  $G$  we consider  $L_K(\alpha \otimes \rho|_H)$ . By properties of Artin L-functions this function has a pole at  $s = 1$  of order  $(\alpha \otimes \rho|_H, 1)_H$ .

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$$(\rho|_H, \bar{\alpha})_H = (\rho, \text{Ind}_H^G(\bar{\alpha}))_G.$$

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It means that equality  $L_K(\alpha \otimes \rho_i|_H) = L_{K'}(\alpha' \otimes \rho_i|_{H'})$  implies

$$(\rho_i, \text{Ind}_H^G(\bar{\alpha}))_G = (\rho_i, \text{Ind}_{H'}^G(\bar{\alpha}'))_G.$$

Since  $\rho_i$  runs over all irreducible representations of  $G$  it means that

$$\text{Ind}_H^G(\bar{\alpha}) = \text{Ind}_{H'}^G(\bar{\alpha}')$$

and therefore  $\text{Ind}_H^G(\alpha) = \text{Ind}_{H'}^G(\alpha')$ .

**From (3) to (1).** By the Frobenius reciprocity for each  $i$ ,  $j \in \{1 \dots n\}$  we have:

$$\begin{aligned}(\mathrm{Ind}_H^G(\alpha \otimes \rho_i|_H), \rho_j)_G &= (\alpha \otimes \rho_i|_H, \rho_j|_H)_H = (\alpha, (\bar{\rho}_i \otimes \rho_j)|_H)_H = \\ &= (\mathrm{Ind}_H^G(\alpha), \bar{\rho}_i \otimes \rho_j)_G,\end{aligned}$$

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By our assumptions  $\text{Ind}_H^G(\alpha) = \text{Ind}_{H'}^G(\alpha')$ , therefore we have:

$$(\text{Ind}_H^G(\alpha), \bar{\rho}_i \otimes \rho_j)_G = (\text{Ind}_{H'}^G(\alpha'), \bar{\rho}_i \otimes \rho_j)_G,$$

and hence for each irreducible representation  $\rho_i$ , we have:

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Finally, by the Artin induction property we have that:

$$L_K(\alpha \otimes \rho_i|_H) = L_{\mathbb{F}_q(x)}(\text{Ind}_H^G(\alpha \otimes \rho_i|_H)),$$

and therefore we are done.

# Applications

Pick  $\alpha$  and  $\alpha'$  to be trivial representations. Then  $K$  and  $K'$  are arithmetically equivalent if and only if  $L_K(\rho|_H) = L_{K'}(\rho|_{H'})$  for all complex irreducible representations  $\rho$  of  $G$ . In particular we have the following result:

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## Theorem

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*then any Gassmann triple  $(G, H, H')$  is trivial and therefore the list  $L_K(\rho)$  determines the function field  $K$  up to isomorphism as extensions of  $\mathbb{F}_q(x)$ .*

# Examples of Gassmann Triples

In order to find examples of arithmetically equivalent function fields we must find a non-trivial example of a Gassmann triple  $(G, H, H')$  and solve the inverse Galois problem for  $G$ . Gassmann triples corresponding to field extensions of degree up to 15 were classified by Bart de Smit.

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It follows that fields with Galois group  $G = \mathrm{PGL}_3(\mathbb{F}_2) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$  give rise to at least two non-trivial Gassmann triples: one in degree seven and one in degree fourteen. Also, fields with Galois group  $G = \mathrm{PSL}_2(\mathbb{F}_{11})$  give rise to at least one pair of arithmetically equivalent fields of degree eleven.

## Example

Let  $p = 7$ ,  $q = p^2$  and let  $\alpha$  be a generator of  $\mathbb{F}_q^*$ . Consider the function field extension of  $\mathbb{F}_q(x)$  given by  $f(y) = y^{p+1} + y - x^{p+1}$ . It's splitting field  $N$  has degree 168 and Galois group  $\text{Gal}(N : \mathbb{F}_q(x)) \simeq \text{PGL}_3(\mathbb{F}_2)$ . Inside this field we have at least two pairs of arithmetically equivalent global function fields:

- 1  $K_1 : y^7 + 5x^8y^3 + \alpha^4x^{12}y + 6$  and  
 $K'_1 : y^7 + 5x^8y^3 + \alpha^{28}x^{12}y + 6;$
- 2  $K_2 : y^{14} + 3x^8y^6 + \alpha^4x^{12}y^2 + 5$  and  
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**Problem:**  $K_1 \simeq K'_1$ ,  $K_2 \simeq K'_2$  as abstract fields. In other words corresponding curves  $X$  and  $X'$  are isomorphic.

# Examples of Gassmann Triples

**Solution:** we need to find a family of arithmetically equivalent fields and then implement a computer search, say in Magma. We will use torsion points on elliptic curves defined over  $\mathbb{F}_q(t)$ .

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Suppose that  $p > 3$  and fix prime  $l \neq p$  and two parameters  $a, b$  in  $\mathbb{F}_q(t)$ . Consider an ordinary elliptic curve  $E$  over  $\mathbb{F}_q(t)$  defined by  $y^2 = x^3 + ax + b$  with non-constant  $j$ -invariant. Consider the field extension  $K$  of  $\mathbb{F}_q(t)$  by adding the coordinates of all points of  $E[l]$ . Let  $N$  denotes the separable closure  $N$  of  $K$  and  $G$  denotes the Galois group  $\text{Gal}(N : \mathbb{F}_q(t))$ . Let  $H$  be the subgroup of  $\mathbb{F}_l^\times$  generated by  $q$ .

# Examples of Gassmann Triples

The analogue of the so-called *Serre's open image theorem* for function fields proved by Igusa in 1959 states that for big enough  $l$  depending on  $q$  we have the following exact sequence:

$$1 \rightarrow \mathrm{SL}_2(\mathbb{F}_l) \rightarrow G \rightarrow H \rightarrow 1.$$

Here  $H$  corresponds to the constant field extension and  $\mathrm{SL}_2(\mathbb{F}_l)$  to the geometric extension. In particular if  $q = 1 \pmod{l}$  then we obtain geometric extension with galois group  $\mathrm{SL}_2(\mathbb{F}_l)$  and therefore also  $\mathrm{PSL}_2(\mathbb{F}_l)$  as quotient group.

# Examples of Gassmann Triples

## Example

*In the above settings let  $p = 29$  and  $l = 7$ ,  $a = t$ ,  $b = t + 1$ . It gives us the geometric extension  $K/\mathbb{F}_{29}(t)$  with the Galois group isomorphic to  $\mathrm{PSL}_2(\mathbb{F}_7)$ . Inside this normal closure following two arithmetically equivalent fields are not geometrically isomorphic:*

$$\begin{aligned} &x^7 + 20tx^6 + 14t^2x^5 + (6t^3 + 11t^2 + 22t + 11)x^4 + (5t^4 + 23t^3 + 17t^2 + 23t)x^3 + \\ &+ (20t^5 + 13t^4 + 26t^3 + 13t^2)x^2 + (5t^6 + 20t^5 + 5t^3 + 21t^2 + 14t + 18)x + \\ &\quad + 23t^7 + 26t^6 + 19t^5 + 10t^4 + 5t^3 + 13t^2 + 25t \end{aligned}$$

*and*

$$\begin{aligned} &x^7 + 16tx^6 + 2t^2x^5 + (18t^3 + 10t^2 + 20t + 10)x^4 + (27t^4 + 3t^3 + 6t^2 + 3t)x^3 + \\ &+ (27t^5 + 17t^4 + 5t^3 + 17t^2)x^2 + (t^6 + 7t^5 + 16t^4 + 15t^3 + 12t^2 + 8t + 2)x + \\ &\quad + 28t^7 + t^6 + 2t^5 + t^4; \end{aligned}$$

## Some Properties of arithmetically equivalent function fields:

- 1 In the number field case we have the following implications:  
 $A_K \simeq A_L \Rightarrow \zeta_K = \zeta_L \iff K \text{ and } L \text{ arithmetically equivalent.}$   
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- 2 Since arithmetically equivalent fields sharing the same zeta-function there exists an isogeny from  $\text{Jac}(X)$  to  $\text{Jac}(X')$ ;
- 3 Since  $h_K = f(1)$  we have that the class-numbers are the same, but class groups may be different. But, one proves that for each Gassmann triple  $(G, H, H')$ , there exists a natural number  $\nu$  such that for each  $l$  co-prime to  $\nu$  we have  $\text{Cl}_l(K) \simeq \text{Cl}_l(K')$  if  $K$  and  $K'$  comes from Gassmann triple  $(G, H, H')$ . For example, for  $G \simeq \text{PSL}_2(\mathbb{F}_7)$  we have  $\nu = 8$ .

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- 4 Proving that fields are not geometrically isomorphic is tricky. For example stated above one checks that actually those function fields correspond to elliptic curves with different  $j$ -invariants.

## Theorem

For a given pair  $K$  and  $K'$  of finite separable geometric extensions of  $F = \mathbb{F}_q(t)$  there exists a Galois extension  $M$  of  $\mathbb{F}_q(t)$  with Galois group  $\tilde{G}$ , such that  $K = M^{\tilde{H}}$  and  $K' = M^{\tilde{H}'}$  for some subgroups  $\tilde{H}, \tilde{H}'$  of  $\tilde{G}$  with the following properties. There exists an abelian character  $\alpha$  of  $\tilde{H}$  such that for any abelian character  $\alpha'$  of  $\tilde{H}'$  the following are equivalent :

- 1 For any irreducible representation  $\rho$  of  $\tilde{G}$  we have equality of Artin  $L$ -functions:  $L_K(\alpha \otimes \rho|_{\tilde{H}}) = L_{K'}(\alpha' \otimes \rho|_{\tilde{H}'})$ ;
- 2  $L_K(\bar{\alpha} \otimes (\psi|_{\tilde{H}})) = L_{K'}(\bar{\alpha}' \otimes (\psi|_{\tilde{H}'}))$ , and  $L_K(\bar{\alpha} \otimes (\psi'|_{\tilde{H}})) = L_{K'}(\bar{\alpha}' \otimes (\psi'|_{\tilde{H}'}))$ , where  $\psi = \text{Ind}_{\tilde{H}}^{\tilde{G}}(\alpha)$  and  $\psi' = \text{Ind}_{\tilde{H}'}^{\tilde{G}}(\alpha')$ ;
- 3 Induced representations  $\psi$  and  $\psi'$  are isomorphic.

Moreover, if those conditions hold then  $K$  and  $K'$  isomorphic as extensions of  $\mathbb{F}_q(t)$ .

# On Monomial Representations

Let  $G$  be a finite group and  $H$  a subgroup. Let  $\chi$  be a one-dimensional representation of  $H$ . Consider the induced representation  $\psi$  of  $G$ :  $\psi = \text{Ind}_H^G \chi$ . By definition  $\psi$  acts on the vector space  $V$  which could be associated with the direct sum of lines  $\bigoplus \mathbb{C}_{g_i}$  where each  $\mathbb{C}_{g_i}$  corresponds to the  $i$ -th left coset  $G/H$ . Such a pair  $(\psi, \bigoplus \mathbb{C}_{g_i})$  is called a *monomial representation*. Let  $H'$  be another subgroup of  $G$  and  $\psi' = \text{Ind}_{H'}^G \chi'$  for one-dimensional  $\chi'$  of  $H'$ . We will say that we have morphism of pairs  $(\psi, \bigoplus \mathbb{C}_{g_i}), (\psi', \bigoplus \mathbb{C}_{g'_j})$  if we have a morphism of representations  $f: \psi \rightarrow \psi'$  such that for each line  $\mathbb{C}_{g_i}$  we have  $f(\mathbb{C}_{g_i}) \subset \mathbb{C}_{g'_j}$  for some  $j$ .

## Lemma

*Suppose we have an isomorphism of monomial representations  $(\psi, \oplus \mathbb{C}_{g_i}) = (\psi', \oplus \mathbb{C}_{g'_j})$ . Then  $H$  is a conjugate of  $H'$  in  $G$ .*

## Example

*On Monomial Representations Let  $G$  be a group of multiplicative quaternions with generators  $a$  and  $b$ . Consider the subgroups  $H_a = \{1, a, -1, -a\}$  and  $H_b = \{1, b, -1, -b\}$ . Let  $\chi_a$  be an isomorphism  $H_a \simeq \mu_4^*$  sending element  $a$  to  $i$ . Let  $\chi_b$  be the same character for  $H_b$ . Then one has  $\text{Ind}_{H_a}^G \chi_a \simeq \text{Ind}_{H_b}^G \chi_b$  as representations, but not as monomial representations.*

# On Monomial Representations

**Settings:** let  $G$  be a finite group and  $H$  a subgroup of index  $n$  and  $C_l = \mu_l$  be a cyclic group of order  $l$ , where  $l$  is an odd prime. Let us consider semi-direct products  $\tilde{G} = C_l^n \rtimes G$  and  $\tilde{H} = C_l^n \rtimes H$ , where  $G$  acts on  $C_l^n$  by permuting its component as cosets  $G/H$ . Let  $g_1, \dots, g_n$  be representatives of left cosets  $G = \cup_i g_i H$ . Without loss of generality we assume that  $g_1 = e$  is the identity element. We define  $\chi$  to be the homomorphism from  $\tilde{H} \rightarrow \mu_l$ , sending an element  $(c_1, \dots, c_n, g)$  to  $c_1$ . This is indeed a homomorphism, since  $H$  fixes the first coset. Then the following is true:

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## Theorem (Bart de Smit)

*For any subgroup  $\tilde{H}' \subset \tilde{G}$  and any abelian character  $\chi' : \tilde{H}' \rightarrow \mathbb{C}^*$  if  $\text{Ind}_{\tilde{H}'}^{\tilde{G}}(\chi') = \text{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)$  then  $\tilde{H}'$  and  $\tilde{H}$  are conjugate in  $\tilde{G}$ .*

# Proof of the Theorem

**Step 1.** Let  $g_1, \dots, g_n$  be a representatives of cosets  $G/H$  with  $g_1$  equals to the identity element. Note that  $g_i$  for  $i \neq 1$  cannot fix the first coset. Consider cosets  $\tilde{G}/\tilde{H}$ . We claim that each such coset for  $i > 1$  could be represented as  $\gamma_i = (1, 1, \dots, 1, g_i)$ , where  $g_i \in G/H$ . This is true since elements of the form  $(\zeta_1, \zeta_2, \dots, \zeta_n, 1)$  are in  $\tilde{H}$ , where  $(\zeta_1, \zeta_2, \dots, \zeta_n) \in C_I^n$ .

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**Step 2.** Let us consider element  $\alpha = (\zeta, 1, \dots, 1, \dots, 1) \in \tilde{H}$  where  $\zeta \in \mu_l$ ,  $\zeta \neq 1$  is in the first position. Such element fixes each coset  $\gamma_i \tilde{H}$ . Therefore if  $\psi = \text{Ind}_{\tilde{H}}^{\tilde{G}}(\chi)$  then  $\psi(\alpha)$  is a diagonal matrix with  $l$ -th roots of unity on the diagonal. Moreover, it is the matrix with the first element is  $\zeta$  on the diagonal and each other diagonal element equals to one. Indeed, by definition of induced representation on the  $i$ -th position we have  $\chi(\gamma_i^{-1} \alpha \gamma_i)$  and it is easy to see that  $\gamma_i^{-1} \alpha \gamma_i$  has first 1 on the first position, provided  $i \neq 1$ .

# Proof of the Theorem

**Step 3.** We claim that  $\psi'(\alpha_i)$  is also a diagonal matrix, where  $\psi' = \text{Ind}_{\tilde{H}'}^{\tilde{G}}(\chi')$ . We know that this is a matrix with exactly one non-zero element in each row and column. Suppose it is not a diagonal, therefore it changes at least two elements and hence trace of this matrix is  $\sum_{k=1}^{n-2} \zeta_i$ , where  $\zeta_i$  are roots of unity. Since  $\psi = \psi'$  we have  $n - 1 + \zeta = \sum_{k=1}^{n-2} \zeta_i$ , which can't be true since the absolute value of the left hand side is strictly bigger than  $n - 2$ . Here we use the fact that  $l > 2$  and therefore  $\zeta \neq \pm 1$ .

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**Step 4.** Let  $A$  be an isomorphism of representations  $\psi$  and  $\psi'$ . We will show that it is *an isomorphism of monomial representations*  $(\psi, \oplus \mathbb{C}_i) = (\psi', \oplus \mathbb{C}_j)$ . Indeed, it suffices to show that in the given basis  $A$  is written as permutation matrix. Suppose it is not and therefore we have at least two non-zero elements in one column. Also it has another non-zero element in some of those two rows, otherwise  $\det(A)$  must be zero which is not since  $A$  is an isomorphism. We have  $A\psi(\alpha) = \psi'(\alpha)A$  which is easy to calculate since  $\psi(\alpha)$  and  $\psi'(\alpha)$  are diagonal and gets the contradiction.

**Thank you!**